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Phase Transition for Ising Frustration Potentials

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A frustration potential is a sum of interactions the terms of which are not simultaneously minimized even in the ground-state spin configurations. Ising models with such potentials can be discussed by the use of contours. The Peierls condition for the phase transition can be properly generalized, taking into account the presence of zero-energy contours. Frustration has some special features in two dimensions, which we study in detail. The connection with models of spin-glasses is discussed.

KEY WORDS: Ising; frustration; contour; phase transition; spin-glass.

1. INTRODUCTION

The Peierls argument[1] for the existence of a phase transition was originally elaborated for the two-dimensional Ising model; however, it proved to be very useful in many other cases. The most successful generalization was made by Pirogov and Sinai[2] a few years ago. By applying a suitably modified definition for the contours which played a key role in the Peierls argument, they proved the Gibbs phase rule for a large class of classical lattice systems. Their results follow from the so-called Peierls condition imposed on the contours, requiring that the energy of a contour is proportional to the measure of its extension over the lattice (its length, so to speak). In order to carry through their treatment they had to confine themselves to the study of models with periodic, finite-range potentials and a finite number of periodic ground states. However, it is easy to construct interactions for classical finite-component lattice gases (spins taking up a finite number of different values) to which there exist infinitely many ground states. For instance, such a model emerged in the Reggeon field theory and was proven to undergo a phase transition.[3] There is also a wide family of lattice systems called "frustration

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models" in the present physics literature[4,5]; we are going to discuss these in this work.

Consider a lattice $\mathbb{Z}$ with Ising spin configurations $s \in \mathscr{R}$, $s: \mathbb{Z} \to \{-1, 1\}$. Suppose we are given a nearest neighbor potential with interactions having the same absolute value but varying in sign:

$$H(s) = -\sum_{|x-y|=1} J_{xy} s(x) s(y) \tag{1.1}$$

One may assume that $|J_{xy}| \equiv 1$ for nearest neighbor pairs. The choice $J_{xy} \equiv 1$ corresponds to the ferromagnet; this and any other interaction having the form

$$J_{xy} = s^0(x) s^0(y) \tag{1.2}$$

with some $s^0 \in \mathscr{R}$ defines a phase transition model: there are two ground states, s^0 and $-s^0$, to which there belong different phases at low temperatures. If $\mathbb{Z}$ is the two or higher dimensional simple cubic lattice, then the antiferromagnetic potential ($J_{xy} \equiv -1$) can be given in the form (1.2); it cannot if the lattice is closed-packed, such as the plane triangular or the fcc one. Whatever $\mathbb{Z}$ is, one always finds infinitely many potentials which cannot be written as (1.2). If Eq. (1.2) fails to hold for any $s^0 \in \mathscr{R}$, then $J_{xy} s(x) s(y) \not\equiv 1$ also for any $s \in \mathscr{R}$. This implies that even in the ground states the energy of some of the bonds is at its higher value. The "frustration" is that of the bonds not able to minimize their energy.

It is easy to imagine that the deviation from (1.2) may lead to the appearance of two ground states s^1 and s^2 differing only in finite noninteracting sets of sites. If the contours of any $s \in \mathscr{R}$ with respect to s^1 are defined as surfaces separating the regions where $s = s^1$ from those where $s = -s^1$, then we realize that s^2 is a configuration having finite contours the creation of which costs no energy. Such—let us say—zero-energy contours trivially do not satisfy the Peierls condition; they may be present in the spin configurations in sufficient numbers to destroy the possibility of a phase transition. We discuss this problem in two theorems. Suppose we know that with respect to a given ground state the zero-energy contours cannot be arbitrarily long and that above some length, contours satisfy the Peierls condition. Then the first theorem asserts the existence of a phase transition. It might occur that the condition imposed on the contours in this theorem is satisfied in a way that there are zero-energy contours around each site. This is hardly possible in two dimensions, our second theorem claims.

After giving the necessary definitions and proving a preparatory Proposition, we formulate these theorems rigorously at the end of Section 2. The proof of Theorem 1 is contained in Section 3; Theorem 2 with three Lemmas is proved in Section 4. A brief discussion is left to Section 5.

2. DEFINITIONS, NOTATIONS, AND THE FORMULATION OF THE RESULTS

In our study the lattice is the ($d \geqslant 2$)-dimensional simple cubic lattice $\mathbb{Z}^d$; the extension to other types of lattices needs only a slight modification. x and $y \in \mathbb{Z}^d$ are nearest neighbors if their Euclidean distance is 1. A pair $\{x, y\} \subset \mathbb{Z}^d$ of nearest neighbors is an edge and is denoted by $\langle xy \rangle$. The whole set of edges of $\mathbb{Z}^d$ is Q. If V, $W \subset \mathbb{Z}^d$, then

$$\langle V, W \rangle = \{\langle xy \rangle \in Q: \quad x \in V, y \in W\}$$

$d(V, W)$ is the Euclidean distance of V and $W \subset \mathbb{Z}^d$. We sometimes use the norm

$$\|x\| = \sum_{i=1}^{d} |x_i|$$

V and W are p-connected if $d(V, W) \leqslant p$; and V and W are connected if they are 1-connected. V is p-connected if V_1 and V_2 are p-connected in any decomposition $V = V_1 \cup V_2$; and V is connected if it is 1-connected. The border of V is $\partial(V) = \langle V, \mathbb{Z}^d - V \rangle$. For any $A \subset Q$ the internal region of A, not necessarily different from the empty set, is

$$\operatorname{Int} A = \bigcup_{\substack{V \subset \mathbb{Z}^d; \partial(V) \subset A \\ V \text{ finite}}} V$$

The potential is meaningful if it is restricted to some finite volume $V \subset \mathbb{Z}^d$:

$$H_V(s) = -\sum_{\langle xy \rangle \in \langle V, \mathbb{Z}^d \rangle} J_{xy} s(x) s(y) \tag{2.1}$$

Let $\mathscr{R}(V, s^0) = \{s \in \mathscr{R}: s = s^0 \text{ on } \mathbb{Z}^d - V\}$. An $s^0 \in \mathscr{R}$ is a ground state in V if $H_V(s^0) \leqslant H_V(s)$ for any $s \in \mathscr{R}(V, s^0)$. The state s^0 is a ground state if it is a ground state in any finite $V \subset \mathbb{Z}^d$.

The border of a configuration s is

$$\Omega(s) = \{\langle xy \rangle \in Q: \quad J_{xy} s(x) s(y) = -1\}$$

Ω is a border if $\Omega = \Omega(s)$ for some $s \in \mathscr{R}$. Suppose that $x, y, z, v \in \mathbb{Z}^d$ form four edges: $\langle xy \rangle$, $\langle yz \rangle$, $\langle zv \rangle$, and $\langle vx \rangle$. The set of these edges is a square,

$$\langle xyzv \rangle = \{\langle xy \rangle, \langle yz \rangle, \langle zv \rangle, \langle vx \rangle\}$$

(In the language of gauge theory, $\langle xyzv \rangle$ is a "plaquette." When dealing with other types of lattices, squares are to be replaced by the corresponding plaquettes.) $\mathscr{F}$ is the set of all squares in $\mathbb{Z}^d$. The termination of a set $A \subset Q$ is

$$\varphi(A) = \{\sigma \in \mathscr{F}: \quad |\sigma \cap A| = 1 \text{ or } 3\}$$

If $\langle xyzv\rangle \in \varphi(\Omega(s))$, then

$$J_{xy}s(x)s(y)J_{yz}s(y)s(z)J_{zv}s(z)s(v)J_{vx}s(v)s(x) = J_{xy}J_{yz}J_{zv}J_{vx} = -1$$

Therefore the termination of a border depends only on the interaction. Let $[Q]$ denote the set of all subsets of Q. The set of all possible borders is just that $Q_\varphi \subset [Q]$ whose elements have the common termination $\varphi \subset \mathscr{F}$. Let Q_0 be the set of borders of configurations if the potential is ferromagnetic. The termination of any border in Q_0 is the empty set. Therefore, the symmetric difference of any border in Q_φ and any one in Q_0 is again in Q_φ. By group properties, any $\Omega \in Q_\varphi$ can be obtained as

$$\Omega = \Omega_0 \circ \partial \tag{2.2}$$

where Ω_0 is a fixed element of Q_φ and $\partial \in Q_0$ is uniquely determined by Ω; $A \circ B$ is the symmetric difference of A and B.

For $A, B \subset Q$ and A finite let

$$k(A|B) = |A - B| - |A \cap B| \tag{2.3}$$

Then, if $s \in \mathscr{R}(V, s^0)$ and $\Omega(s^0) = \Omega_0$,

$$H_V(s) - H_V(s^0) = 2k(\partial|\Omega_0) \tag{2.4}$$

with $\partial = \Omega(s) \circ \Omega_0$. Hereafter, Ω_0 always denotes the border of s^0, which is chosen to be a ground state; ∂ is always an element of Q_0. Hence, $0 \leqslant k(\partial|\Omega_0) \leqslant |\partial|$.

A and $B \subset Q$ are connected if there exists a $\sigma \in \mathscr{F}$ such that $A \cap \sigma \neq \varnothing$ and $B \cap \sigma \neq \varnothing$. An $A \subset Q$ is connected if A_1 and A_2 are connected in any decomposition $A = A_1 \cup A_2$. An element of Q_0 is a contour if any of its actual parts is not in Q_0. In the following, Γ always denotes a contour. Γ is connected and $|\Gamma \cap \sigma| = 0$ or 2 for any $\sigma \in \mathscr{F}$. Any $\partial \in Q_0$ can be decomposed (though not uniquely) into the union of pairwise nonintersecting contours. $k(\partial|\Omega_0)$ is additive in the contours of the decomposition.

Definition. Given a ground state s^0, $x \in \mathbb{Z}^d$ is a Peierls point if there exist L and $c > 0$ such that

$$k(\Gamma|\Omega_0) \geqslant c|\Gamma| \tag{2.5}$$

if $x \in \operatorname{Int}\Gamma$ and $|\Gamma| \geqslant L$. Now, $\mathscr{P}_{L,c}(s^0)$ is the set of Peierls points for which (2.5) holds if $|\Gamma| \geqslant L$; and

$$\mathscr{P}_L(s^0) = \bigcup_{c>0} \mathscr{P}_{L,c}(s^0); \qquad \mathscr{P}(s^0) = \bigcup_{L \geqslant 2d} \mathscr{P}_L(s^0)$$

is the set of all Peierls points.

Let

$$L(x) = \min_{x \in \mathscr{P}_L(s^0)} L \tag{2.6}$$

If $L(x) > 2d$, one can find a contour such that $x \in \operatorname{Int}\Gamma$ and $k(\Gamma|\Omega_0) = 0$. This we call a zero-energy contour.

Proposition. $\mathscr{P}(s^0)$ is either the empty set or it is the whole lattice. If $x \in \mathscr{P}_{L,c}(s^0)$ and

$$|\Gamma| \geqslant \max\left\{L + 2d - 2, \frac{(1-c)(2d-2)}{c}\rho(\Gamma, x) + 4d + \frac{2}{c} - 3\right\}$$

then $k(\Gamma|\Omega_0) > 0$. Here

$$\rho(\Gamma, x) = \min_{y \in \operatorname{Int}\Gamma} \|y - x\|$$

Proof. If $x \in \operatorname{Int}\Gamma$ and $|\Gamma| \geqslant L$, then $k(\Gamma|\Omega_0) > 0$. Suppose that $x \notin \operatorname{Int}\Gamma$ and put $\rho = \rho(\Gamma, x)$. One can find $\delta_1, \ldots, \delta_\rho$ contours such that $|\delta_i| = 2d$ and $\Gamma' = \delta_1 \circ \cdots \circ \delta_\rho \circ \Gamma$ is a contour around x: $x \in \operatorname{Int}\Gamma'$. We have $|\Gamma'| \geqslant |\Gamma| + 2(\rho - 2)(d-1)$ and by using the identity

$$k(\partial_1 \circ \partial_2|\Omega) = k(\partial_1|\Omega) + k(\partial_2|\Omega \circ \partial_1) \tag{2.7}$$

one finds that $k(\Gamma'|\Omega_0) \leqslant k(\Gamma|\Omega_0) + 2\rho(d-1) + 2$. The combination of these estimates yields

$$k(\Gamma|\Omega_0) \geqslant \epsilon|\Gamma|$$

if

$$|\Gamma| \geqslant \max\left\{L - 2(\rho-2)(d-1), 2\frac{\rho(d-1) + 1 - c(\rho-2)(d-1)}{c - \varepsilon}\right\}$$

which proves the Proposition.

From the Proposition it immediately follows that $L(y) \leqslant \text{const} \times \|y\|$.

The original Peierls argument[1] proves the existence of a phase transition if $L(x) = 2d$ for some $x \in \mathbb{Z}^d$. A generalization can be found in the following result:

Theorem 1. In $\mathbb{Z}^d$, let us have a potential of the type (1.1). Assume that one can find a ground state s^0 for which $\mathscr{P}(s^0)$ is not empty. Then there exists a $\beta_0 > 0$ such that a phase transition occurs for inverse temperatures $\beta > \beta_0$.

The most interesting situation described by this theorem—and actually the only case when the original Peierls argument does not work—is that where the sites are all Peierls points but there are zero-energy contours around each site, i.e., $\mathscr{P}(s^0) = \mathbb{Z}^d$ but $\mathscr{P}_{2d}(s^0) = \varnothing$. The following theorem shows that in two dimensions the usual case of a phase transition is somehow the opposite: $\mathscr{P}_{2d}(s^0)$ is an infinite set.

Theorem 2. In two dimensions, consider a potential (1.1) and let s^0 be a ground state with regard to it. Suppose that $\mathscr{P}(s^0)$ is not empty and there exists a $\vartheta > 0$ and a trajectory

$$X = \{x_0 = 0, x_1, x_2, \ldots: \quad x_i \in \mathbb{Z}^2, \quad d(x_i, x_{i-1}) = 1, \quad |x_i| \to \infty\}$$

such that $|\Gamma| < \vartheta$ for any zero-energy contour Γ intersecting with $\langle X, X\rangle$. Then $\mathscr{P}_4(s^0)$ has an infinite connected part. Moreover, $\mathscr{P}_4(s^0)$ is a set of positive density: if $T_K = \{x \in \mathbb{Z}^2: \|x\| \leqslant K\}$, then

$$\liminf_{K \to \infty} \frac{|\mathscr{P}_4(s^0) \cap T_K|}{|T_K|} > 0$$

We note that the conditions of Theorem 2 are met if s^0 is a periodic ground state and $\mathscr{P}(s^0)$ is not empty.

3. THE PROOF OF THEOREM 1

According to the Proposition, $0 \in \mathscr{P}_L(s^0)$ with some $L \geqslant 2d$. We choose a finite $T \subset \mathbb{Z}^d$ such that $0 \in T$ and, if Γ is a contour with the property $\text{Int}\,\Gamma \cap \{0\} \cap \Delta T \neq \varnothing$ then $|\Gamma| \geqslant L$. Here ΔT denotes the set $\{x \in T: d(x, \mathbb{Z}^d - T) = 1\}$. For any finite $V \supset T$ let us consider the restriction of those ground states with regard to T that are equal to s^0 outside V. This set of configurations on T is denoted by $G_V(s^0, T)$. Plainly, $G_V(s^0, T) \subset G_{V'}(s^0, T)$ if $V \subset V'$. Moreover, $|G_V(s^0, T)| \leqslant 2^{|T|}$, whence there is a finite $V_0 \subset \mathbb{Z}^d$ such that $G_V(s^0, T) = G(s^0, T)$ independently of V for any $V \supset V_0$. The proof of the theorem is given in two steps.

(i) $G(s^0, T) \cap G(-s^0, T) = \varnothing$. Suppose that this is not true and let $s_T \in G(s^0, T) \cap G(-s^0, T)$. Then there exist two ground states s^1 and s^3 such that $s^1 = s^3 = s_T$ on T and $s^1 = -s^3 = s^0$ on $\mathbb{Z}^d - V$. Moreover, $s^2 \equiv -s^3$ is also a ground state. Let $\partial_i = \Omega(s^i) \circ \Omega_0$ for $i = 1, 2$ and $\partial_{12} = \Omega(s^1) \circ \Omega(s^2)$; then $k(\partial_i | \Omega_0) = 0$. Take the following decomposition of ∂_1 into nonintersecting terms:

$$\partial_1 = \bigcup_{j=1}^{n} \Gamma_j \cup \partial'$$

chosen in such a way that $0 \in \text{Int}\,\Gamma_j$ for $j = 1, 2, \ldots, n$ and $0 \notin \text{Int}\,\partial'$. According to our choice of T, $\text{Int}\,\Gamma_j \cap \Delta T = \varnothing$ and therefore $\bigcup_{j=1}^n \Gamma_j \subset \langle T, T\rangle$. On the other hand, $\partial_{12} \subset \langle V - T, \mathbb{Z}^d\rangle$ and hence we obtain $\partial_{12} \cap (\bigcup_{j=1}^n \Gamma_j) = \varnothing$. Let

$$X = \{x_0 = 0, x_1, x_2, \ldots: \quad d(x_i, x_{i-1}) = 1, \quad |x_i| \to \infty\} \subset \mathbb{Z}^d$$

be any trajectory. Then $|\partial' \cap \langle X, X\rangle|$ is even and $|\partial_{12} \cap \langle X, X\rangle|$ is odd: the first follows from $0 \notin \text{Int}\,\partial'$ and the second from the fact that any decomposition of ∂_{12} into the union of nonintersecting contours must have an odd number of contours encircling the whole T. Then one obtains that

$$\begin{aligned}&|(\partial_{12} \circ \partial') \cap \langle X, X\rangle|\\ &\quad = |\partial_{12} \cap \langle X, X\rangle \circ \partial' \cap \langle X, X\rangle|\\ &\quad = |\partial_{12} \cap \langle X, X\rangle| + |\partial' \cap \langle X, X\rangle| - 2|\partial_{12} \cap \partial' \cap \langle X, X\rangle|\end{aligned}$$

is also an odd number. Therefore, there is at least one $\Gamma \subset \partial_{12} \circ \partial'$ such that $0 \in \operatorname{Int} \Gamma$. Furthermore, $\Gamma \cap \partial_{12} \neq \varnothing$ and hence $\operatorname{Int} \Gamma \cap \Delta T \neq \varnothing$; this means that $k(\Gamma|\Omega_0) > 0$ and $k(\partial_{12} \circ \partial'|\Omega_0) > 0$. This is, however, a contradiction, because from $\partial_2 = \partial_1 \circ \partial_{12}$ one gets

$$k(\partial_2|\Omega_0) = \sum_{j=1}^{n} k(\Gamma_j|\Omega_0) + k(\partial_{12} \circ \partial'|\Omega_0)$$

which is a sum of nonnegative terms and is equal to zero.

(ii) Let $V \supset V_0$ and let $\mu_{\beta,s^0,V}$ be the Gibbs probability measure on $\mathscr{R}(V, s^0)$ at inverse temperature β. Then by the use of Eq. (2.4) we get

$$\mu_{\beta,s^0,V}(s) = Z(\beta, s^0, V)^{-1} \exp[-2\beta k(\partial|\Omega_0)] \tag{3.1}$$

if $s \in \mathscr{R}(V, s^0)$ and $\partial = \Omega(s) \circ \Omega_0$. Suppose that for some $\epsilon < \frac{1}{2}$ we can prove the existence of a $\beta_0(\epsilon)$ such that

$$\mu_{\beta,s^0,V}(s_T \notin G(s^0, T)) < \epsilon \tag{3.2}$$

if $\beta > \beta_0(\epsilon)$. Here s_T is the restriction of $s \in \mathscr{R}(V, s^0)$ to T. Then

$$\mu_{\beta,s^0,V}(s_T \in G(s^0, T)) > 1 - \epsilon \tag{3.3}$$

and

$$\begin{aligned}\mu_{\beta,-s^0,V}(s_T \in G(s^0, T)) &= \mu_{\beta,s^0,V}(s_T \in G(-s^0, T))\\ &\leqslant \mu_{\beta,s^0,V}(s_T \notin G(s^0, T)) < \epsilon\end{aligned} \tag{3.4}$$

The first inequality in (3.4) is a consequence of (i). The inequalities (3.3) and (3.4) together prove the existence of a phase transition for $\beta > \beta_0(\epsilon)$. To obtain this result we show, by applying Peierls' argument to each point of T, that (3.2) is true. Let $\partial(s) = \Omega(s) \circ \Omega_0$; then

$$\begin{aligned}&\{s \in \mathscr{R}(V, s^0):\quad s_T \notin G(s^0, T)\}\\ &\quad = \{s \in \mathscr{R}(V, s^0):\quad \partial(s) \supset \Gamma \quad \text{such that } k(\Gamma|\Omega_0) > 0\\ &\qquad\quad \text{and} \quad \operatorname{Int}\Gamma \cap T \neq \varnothing\}\\ &\quad = \bigcup_{y \in T} \quad \bigcup_{\substack{\Gamma \subset \langle V, \mathbb{Z}^d\rangle:\\ y \in \operatorname{Int}\Gamma, k(\Gamma|\Omega_0) > 0}} \{s \in \mathscr{R}(V, s^0):\quad \partial(s) \supset \Gamma\}\end{aligned} \tag{3.5}$$

Let

$$c(y) = \min_{\substack{\Gamma : k(\Gamma|\Omega_0) > 0 \\ y \in \mathrm{Int}\,\Gamma}} \frac{k(\Gamma|\Omega_0)}{|\Gamma|}$$

and put $c = \min_{y \in T} c(y)$. This is positive because every point of T is a Peierls point. For every contour occurring in (3.5), $k(\Gamma|\Omega_0) \geqslant c|\Gamma|$. We write

$$\mu_{\beta,s^0,V}(\Gamma) \equiv \sum_{\substack{s \in \mathscr{R}(V,s^0): \\ \partial(s) \supset \Gamma}} \mu_{\beta,s^0,V}(s)$$

Now the following inequalities hold:

$$\mu_{\beta,s^0,V}(s_T \notin G(s^0, T)) \leqslant \sum_{y \in T} \sum_{\substack{\Gamma : y \in \mathrm{Int}\,\Gamma \\ k(\Gamma|\Omega^0) > 0}} \mu_{\beta,s^0,V}(\Gamma)$$

$$\leqslant \sum_{y \in T} \sum_{\Gamma : y \in \mathrm{Int}\,\Gamma} e^{-2\beta c|\Gamma|} \leqslant |T| \sum_{l \geqslant 2d} l \cdot 3^l e^{-2\beta c l}$$

The upper bound is convergent if $\beta > \log 3/2c$ and goes to zero with increasing β. Therefore, for any $\epsilon < \frac{1}{2}$ one can find a $\beta_0(\epsilon) < \infty$ such that (3.2) holds if $\beta > \beta_0(\epsilon)$.

4. THE PROOF OF THEOREM 2

We begin by proving three lemmas.

Lemma 1. In $\mathbb{Z}^d$, let $\Gamma_1, \ldots, \Gamma_N$ be finite contours. Then

$$\Gamma_1 \circ \Gamma_2 \circ \cdots \circ \Gamma_N = \partial(\mathrm{Int}\,\Gamma_1 \circ \mathrm{Int}\,\Gamma_2 \circ \cdots \circ \mathrm{Int}\,\Gamma_N)$$

for any $N \geqslant 1$.

Proof. For $N = 1$ the statement is a direct consequence of the definitions. We therefore have $\{x, y\} \cap \mathrm{Int}\,\Gamma \neq \varnothing$ and $\{x, y\} - \mathrm{Int}\,\Gamma \neq \varnothing$ if $\langle xy \rangle \in \Gamma$. Now let $N > 1$. Then $\langle xy \rangle \in \Gamma_1 \circ \cdots \circ \Gamma_N$ iff $\langle xy \rangle \in \Gamma_{i_1} \cap \Gamma_{i_2} \cap \cdots \cap \Gamma_{i_m}$ for some odd $m \leqslant N$ and $\langle xy \rangle \notin \Gamma_j$ for any other j. This happens iff

$$\{x, y\} \cap \mathrm{Int}\,\Gamma_{i_k} \neq \varnothing, \qquad \{x, y\} - \mathrm{Int}\,\Gamma_{i_k} \neq \varnothing$$

for $k = 1, 2, \ldots, m$ and either $\{x, y\} \subset \mathrm{Int}\,\Gamma_j$, or $\{x, y\} \subset \mathbb{Z}^d - \mathrm{Int}\,\Gamma_j$ for any other j. Assume now that $x \in \mathrm{Int}\,\Gamma_{i_k}$ holds for exactly n values of $k \in \{1, 2, \ldots, m\}$ and $x \in \mathrm{Int}\,\Gamma^j$ holds for exactly p values of $j \neq i_1, i_2, \ldots, i_m$. Then x is in the internal region of exactly $n + p$ contours. On the other hand, $y \in \mathrm{Int}\,\Gamma_{i_k}$ just for the remaining $m - n$ values of k and $y \in \mathrm{Int}\,\Gamma^j$ for the same j's as x is; altogether, y is inside $m - n + p$ contours. Since $n + p$ and $m - n + p$ have different parity, we can find one and only one of x and y in $\mathrm{Int}\,\Gamma_1 \circ \cdots \circ \mathrm{Int}\,\Gamma_N$ and this is true iff $\langle xy \rangle \in \partial(\mathrm{Int}\,\Gamma_1 \circ \cdots \circ \mathrm{Int}\,\Gamma_N)$.

We say that a set of contours $\Gamma_1, ..., \Gamma_N$ is a minimal ring around the origin if it has the following properties:

$$0 \notin \bigcup_{i=1}^{N} \operatorname{Int} \Gamma_i, \quad 0 \in \operatorname{Int} \bigcup_{i=1}^{N} \Gamma_i, \quad 0 \notin \operatorname{Int} \bigcup_{i \neq j} \Gamma_i \qquad \text{any } j = 1, 2, ..., N$$

In two dimensions, the indices of the contours in a minimal ring can be chosen in such a way that $\Gamma_i \cup \Gamma_{i+1}$ is connected for any $i = 1, 2, ..., N$ $(N + 1 \equiv 1)$. From the minimality property it follows that $\Gamma_i \cup \Gamma_j$ is not connected if $j \neq i - 1, i, i + 1$.

Lemma 2. In $\mathbb{Z}^2$, let $\Gamma_1, ..., \Gamma_N$ be a minimal ring around the origin for some $N \geqslant 3$. Assume that $\operatorname{Int} \Gamma_i \circ \operatorname{Int} \Gamma_{i+1}$ is $\sqrt{2}$-connected for each i; $N + 1 \equiv 1$. Then there exists a $\Gamma \subset \Gamma_1 \circ \cdots \circ \Gamma_N$ such that $0 \in \operatorname{Int} \Gamma$.

Proof. Let Y_i be a trajectory which starts from the origin and

$$Y_i \cap \bigcup_{j=1}^{N} \operatorname{Int} \Gamma_j \subset \operatorname{Int} \Gamma_i - \bigcup_{j \neq i} \operatorname{Int} \Gamma_j$$

The minimality of the contour set assures the existence of such a trajectory for each $i = 1, ..., N$. Now, $Y_i \cup Y_{i+1}$ divides $\mathbb{Z}^2$ into two halves. One, called the $(i, i + 1)$ sector, does not contain any points of $\operatorname{Int} \Gamma_j$ if $j \neq i, i + 1$; $Y_i \cup Y_{i+1}$ is considered to belong here. Any trajectory starting from the origin and proceeding inside the $(i, i + 1)$ sector has an intersection with

$$A_i \equiv \operatorname{Int} \Gamma_i \circ \operatorname{Int} \Gamma_{i+1} - \bigcup_{j \neq i, i+1} \operatorname{Int} \Gamma_j$$

Furthermore, $d(A_i, A_{i+1}) = 0$ because $A_i \cap A_{i+1} \cap Y_{i+1} \neq \varnothing$. Therefore, any trajectory starting from the origin intersects with

$$\bigcup_{i=1}^{N} A_i = \operatorname{Int} \Gamma_1 \circ \cdots \circ \operatorname{Int} \Gamma_N$$

from which it follows that $0 \in \operatorname{Int} \Gamma$ for some

$$\Gamma \subset \partial(\operatorname{Int} \Gamma_1 \circ \cdots \circ \operatorname{Int} \Gamma_N)$$

By Lemma 1, that was just the assertion.

Lemma 3. In $\mathbb{Z}^d$, let Γ_1 and Γ_2 be zero-energy contours and $\operatorname{Int} \Gamma_1 \circ \operatorname{Int} \Gamma_2 = C_1 \cup C_2$, where either $C_2 = \varnothing$ or $d(C_1, C_2) > 1$. Put

$$A_i = C_i \cap \operatorname{Int} \Gamma_1 - \operatorname{Int} \Gamma_2, \qquad B_i = C_i \cap \operatorname{Int} \Gamma_2 - \operatorname{Int} \Gamma_1 \qquad \text{for } i = 1, 2$$

$$I = \operatorname{Int} \Gamma_1 \cap \operatorname{Int} \Gamma_2$$

Then $k(\partial(A_1 \cup I \cup B_2) | \Omega_0) = k(\partial(A_2 \cup I \cup B_1) | \Omega_0) = 0$

Proof. We introduce the notations

$$x_{1i} = \langle A_i, B_i \rangle, \qquad x_{2i} = \langle A_i, \mathbb{Z}^d - (\operatorname{Int} \Gamma_1 \cup \operatorname{Int} \Gamma_2) \rangle$$
$$x_{3i} = \langle I, B_i \rangle, \qquad x_4 = \langle I, \mathbb{Z}^d - (\operatorname{Int} \Gamma_1 \cup \operatorname{Int} \Gamma_2) \rangle$$
$$x_{5i} = \langle \mathbb{Z}^d - (\operatorname{Int} \Gamma_1 \cup \operatorname{Int} \Gamma_2), B_i \rangle, \qquad x_{6i} = \langle A_i, I \rangle$$

These eleven sets ($i = 1, 2$) are pairwise nonintersecting. Using them, we can write

$$\begin{aligned}
\Gamma_1 &= x_{11} \cup x_{12} \cup x_{21} \cup x_{22} \cup x_{31} \cup x_{32} \cup x_4 \\
\Gamma_2 &= x_{11} \cup x_{12} \cup x_4 \cup x_{51} \cup x_{52} \cup x_{61} \cup x_{62} \\
\partial(A_1 \cup I \cup B_2) &= x_{11} \cup x_{12} \cup x_{21} \cup x_{31} \cup x_4 \cup x_{52} \cup x_{62} \\
\partial(A_2 \cup I \cup B_1) &= x_{11} \cup x_{12} \cup x_{22} \cup x_{32} \cup x_4 \cup x_{51} \cup x_{61}
\end{aligned}$$

Turning to the energies, one finds that

$$k(\partial(A_1 \cup I \cup B_2)|\Omega_0) + k(\partial(A_2 \cup I \cup B_1)|\Omega_0) = k(\Gamma_1|\Omega_0) + k(\Gamma_2|\Omega_0) = 0$$

Since $k(\partial|\Omega_0)$ is nonnegative, this proves the assertion.

Using these lemmas, the proof of the theorem is the following.

1. Suppose first that every maximal connected part of $\mathscr{P}_4(s^0)$ is finite.

(i) According to the Proposition, $0 \in \mathscr{P}_{L,c}(s^0)$ with some $L \geqslant 4$ and $c > 0$. Let

$$K = \max\{L + 1, L/2c, \vartheta/2c^2\} \tag{4.1}$$

and $T_K = \{x \in \mathbb{Z}^2 : \|x\| \leqslant K\}$; then one can find a $V \supset T_K$ finite connected set such that

$$\delta V \cap \mathscr{P}_4(s^0) \equiv \{x \in \mathbb{Z}^2 : d(x, V) = 1\} \cap \mathscr{P}_4(s^0) = \varnothing$$

Therefore, a zero-energy contour $\Gamma(x)$ exists around each point $x \in \delta V$. Now, $\|x\| > L$ and δV is $\sqrt{2}$-connected; it follows that $0 \notin \bigcup_{x \in \delta V} \operatorname{Int} \Gamma(x)$ but $0 \in \operatorname{Int} \bigcup_{x \in \delta V} \Gamma(x)$. If $\{\Gamma(x) : x \in \delta V\}$ is not a minimal ring around the origin, it can be turned into such a set by omitting several contours. Let

$$S^{(0)} = \{\Gamma_1^{(0)}, \ldots, \Gamma_{N_0}^{(0)}\} \subset \{\Gamma(x) : \quad x \in \delta V\}$$

be a minimal ring and the indices be chosen such that $\Gamma_i^{(0)} \cup \Gamma_{i+1}^{(0)}$ is connected for $i = 1, \ldots, N_0$ ($N_0 + 1 \equiv 1$).

(ii) Assume there exists an i for which $\operatorname{Int} \Gamma_i^{(0)} \circ \operatorname{Int} \Gamma_{i+1}^{(0)}$ is not $\sqrt{2}$-connected. We apply Lemma 3 with the identification $\Gamma_1 \equiv \Gamma_i^{(0)}, \Gamma_2 \equiv \Gamma_{i+1}^{(0)}$. It is plain that

$$d(\operatorname{Int} \Gamma_{i+1}^{(0)}, \operatorname{Int} \Gamma_{i+2}^{(0)}) \leqslant \sqrt{2}, \qquad d(\operatorname{Int} \Gamma_i^{(0)}, \operatorname{Int} \Gamma_{i+2}^{(0)}) > \sqrt{2}$$

therefore

$$d(\operatorname{Int} \Gamma_i^{(0)} \circ \operatorname{Int} \Gamma_{i+1}^{(0)}, \operatorname{Int} \Gamma_{i+2}^{(0)}) \leqslant \sqrt{2}$$

Now, $\text{Int}\,\Gamma_i^{(0)} \circ \text{Int}\,\Gamma_{i+1}^{(0)} = C_1 \cup C_2$, where C_1 is chosen to be a maximal connected set of the symmetric difference such that $d(C_1, \text{Int}\,\Gamma_{i+2}^{(0)}) \leqslant \sqrt{2}$. The set C_2 is not empty and $d(C_1, C_2) > 1$. Using the notations of Lemma 3, we have $C_1 = A_1 \cup B_1$, where

$$d(A_1, \text{Int}\,\Gamma_{i+2}^{(0)}) > \sqrt{2}, \qquad d(B_1, \text{Int}\,\Gamma_{i+2}^{(0)}) \leqslant \sqrt{2}$$

and, plainly, $B_1 \neq \varnothing$. It follows therefore that $\partial(A_2 \cup I \cup B_1)$ is connected with $\Gamma_{i+2}^{(0)}$ and there is a $\Gamma' \subset \partial(A_2 \cup I \cup B_1)$ contour connected with $\Gamma_{i+2}^{(0)}$ and having zero energy. Moreover, $\Gamma' \neq \Gamma_{i+1}^{(0)}$. For, let us suppose that $\Gamma' = \Gamma_{i+1}^{(0)}$; by definition, $\text{Int}\,\Gamma_{i+1}^{(0)} = B_1 \cup I \cup B_2$ and in this case B_2 must be empty. But $C_2 \neq \varnothing$; therefore $A_2 \neq \varnothing$. Now we have $\Gamma' = \partial(I \cup B_1)$ and $\partial(A_2 \cup I \cup B_1) = \Gamma' \cup \partial(A_2)$, which is a union of nonintersecting terms. This means that $d(I \cup B_1, A_2) > 1$ and hence $d(I, A_2) > 1$. Notice, however, that

$$\text{Int}\,\Gamma_i^{(0)} \cup \text{Int}\,\Gamma_{i+1}^{(0)} = I \cup C_1 \cup C_2 = I \cup C_1 \cup A_2$$

is a connected set because $\Gamma_i^{(0)} \cap \Gamma_{i+1}^{(0)} \neq \varnothing$ [otherwise

$$\Gamma_i^{(0)} \cup \Gamma_{i+1}^{(0)} = \Gamma_i^{(0)} \circ \Gamma_{i+1}^{(0)} = \partial(\text{Int}\,\Gamma_i^{(0)} \circ \text{Int}\,\Gamma_{i+1}^{(0)})$$

is connected, which yields $\text{Int}\,\Gamma_i^{(0)} \circ \text{Int}\,\Gamma_{i+1}^{(0)}$ to be $\sqrt{2}$-connected]. But $d(C_1, A_2) > 1$ and therefore $d(I \cup C_1, A_2) > 1$, contradicting the connectivity. We also have $\Gamma' \neq \Gamma_i^{(0)}$ because $\Gamma_i^{(0)}$ and $\Gamma_{i+2}^{(0)}$ are not connected. It is true that $\Gamma' \subset \Gamma_i^{(0)} \cup \Gamma_{i+1}^{(0)}$ and, contours having no contour subsets, we get $\Gamma' \cap \Gamma_i^{(0)} \neq \varnothing$. The contours of the set $S' \equiv S^{(0)} - \{\Gamma_{i+1}^{(0)}\} \cup \{\Gamma'\}$ have the properties $k(\Gamma|\Omega_0) = 0$ for $\Gamma \in S'$, $0 \notin \bigcup_{\Gamma \in S'} \text{Int}\,\Gamma$, but $0 \in \text{Int} \bigcup_{\Gamma \in S'} \Gamma$ and $\bigcup_{\Gamma \in S'} \Gamma \subset \bigcup_{x \in \partial V} \Gamma(x)$. Let $S^{(1)} \subset S'$ be a minimal ring around the origin.

(iii) Then

$$\Big|\bigcup_{\Gamma \in S^{(1)}} \Gamma\Big| + \Big|\bigcup_{\Gamma \in S^{(1)}} \text{Int}\,\Gamma\Big| < \Big|\bigcup_{\Gamma \in S^{(0)}} \Gamma\Big| + \Big|\bigcup_{\Gamma \in S^{(0)}} \text{Int}\,\Gamma\Big| \tag{4.2}$$

This we prove by showing that

$$\Big|\bigcup_{\Gamma \in S'} \Gamma\Big| + \Big|\bigcup_{\Gamma \in S'} \text{Int}\,\Gamma\Big| < \Big|\bigcup_{\Gamma \in S^{(0)}} \Gamma\Big| + \Big|\bigcup_{\Gamma \in S^{(0)}} \text{Int}\,\Gamma\Big|$$

It is sufficient to show that

$$|\Gamma_{i+1}^{(0)} - \Gamma_i^{(0)} - \Gamma_{i+2}^{(0)} - \Gamma'| + |\text{Int}\,\Gamma_{i+1}^{(0)} - \text{Int}\,\Gamma_i^{(0)} - \text{Int}\,\Gamma_{i+2}^{(0)} - \text{Int}\,\Gamma'| \geqslant 1 \tag{4.3}$$

If $B_2 = \varnothing$, then $C_2 = A_2 \neq \varnothing$. Also, $x_{62} = \langle A_2, I\rangle \neq \varnothing$ because otherwise $d(A_2, I) > 1$, which has been excluded previously (I is not empty, due to the choice of i). But

$$x_{62} \cap (\Gamma_i^{(0)} \cup \Gamma' \cup \Gamma_{i+2}^{(0)}) = \varnothing \qquad \text{and} \qquad x_{62} \subset \Gamma_{i+1}^{(0)}$$

Therefore the first term of (4.3) is positive. If $B_2 \neq \varnothing$, then $d(B_1 \cup I, B_2) = 1$ because of the connectivity of $\text{Int}\,\Gamma_{i+1}^{(0)}$. Furthermore, either $B_1 = \varnothing$ or $d(B_1, B_2) > 1$; in both cases $d(I, B_2) = 1$. It follows that $B_2 \not\subset \text{Int}\,\Gamma_{i+2}^{(0)}$ and, at the same time,

$$B_2 \subset \text{Int}\,\Gamma_{i+1}^{(0)} \qquad \text{and} \qquad B_2 \cap (\text{Int}\,\Gamma_i^{(0)} \cup \text{Int}\,\Gamma') = \varnothing$$

This assures the positivity of the second term of (4.3).

In (ii) we defined a transformation leading from $S^{(k)}$ to a new minimal ring of zero-energy contours $S^{(k+1)}$. Inequality (4.2) guarantees that after a finite number of steps one gets a minimal ring around the origin $S^{(m)} \equiv S = \{\Gamma_1, \ldots, \Gamma_N\}$ formed by zero-energy contours with the property that $\text{Int}\,\Gamma_i \circ \text{Int}\,\Gamma_{i+1}$ is $\sqrt{2}$-connected for $i = 1, 2, \ldots, N$ $(N + 1 \equiv 1)$. Also, $\bigcup_{i=1}^{N} \Gamma_i \subset \bigcup_{x \in \delta V} \Gamma(x)$.

(iv) Let $x \in \mathbb{Z}^2$ and Γ such that $x \in \text{Int}\,\Gamma$, $k(\Gamma|\Omega_0) = 0$, and let $\rho = \min_{y \in \text{Int}\,\Gamma} \|y\|$. Then $\rho \geqslant c[\|x\| - (L + 3)/2]$. This follows from the inequalities

$$2[(\|x\| - \rho) + 1] + 2 \leqslant |\Gamma| < \frac{2(1 - c)}{c}\rho + \frac{2}{c} + L + 7$$

The first is obvious and the second comes from the Proposition. $\rho = \rho(\Gamma, 0)$ with the notations of the Proposition. If $x \in \delta V$, then $\|x\| \geqslant K + 1$ and with the choice (4.1) we get $\rho(\Gamma(x), 0) \geqslant Kc/2$. As a consequence, if $\gamma \subset \bigcup_{x \in \delta V} \Gamma(x)$ is a contour around the origin, then $|\gamma| \geqslant 4(Kc - 1) \geqslant L$. The second inequality is due to the choice of K and the fact that $L \geqslant 4$.

(v) We now complete the proof of the first assertion of the theorem. $|S| = N \geqslant 2$; suppose first that $N = 2m, m \geqslant 2$. Let $\partial_e = \Gamma_2 \circ \Gamma_4 \circ \cdots \circ \Gamma_N$ and $\partial_o = \Gamma_1 \circ \Gamma_3 \circ \cdots \circ \Gamma_{N-1}$; plainly $k(\partial_e|\Omega_0) = k(\partial_o|\Omega_0) = 0$. Also,

$$k(\partial_e \circ \partial_o|\Omega_0 \circ \partial_e) = k(\partial_o|\Omega_0) - k(\partial_e|\Omega_0) = 0$$

where we applied Eq. (2.7). Therefore, if $\gamma \subset \partial_e \circ \partial_o$ is a contour around the origin—and such a contour certainly exists, due to Lemma 2—then $k(\gamma|\Omega_0 \circ \partial_e) = 0$. Consider now the border $\gamma \circ \partial_e$. We have $0 \in \text{Int}\,\gamma$ and $0 \notin \text{Int}\,\partial_e$; therefore

$$0 \in \text{Int}\,\gamma \circ \text{Int}\,\partial_e = \text{Int}\,\gamma \circ \text{Int}\,\Gamma_2 \circ \text{Int}\,\Gamma_4 \circ \cdots \circ \text{Int}\,\Gamma_N$$

This means that there exists a contour γ' around the origin such that

$$\gamma' \subset \gamma \circ \partial_e = \partial(\text{Int}\,\gamma \circ \text{Int}\,\Gamma_2 \circ \cdots \circ \text{Int}\,\Gamma_N).$$

Whence,

$$0 \leqslant k(\gamma'|\Omega_0) \leqslant k(\gamma \circ \partial_e|\Omega_0) = k(\partial_e|\Omega_0) + k(\gamma|\Omega_0 \circ \partial_e) = 0$$

i.e., $k(\gamma'|\Omega_0) = 0$. However, $\gamma' \subset \bigcup_{x \in \delta V} \Gamma(x)$ and from (iv) one knows that $|\gamma'| \geqslant L$. This contradicts $0 \in \mathscr{P}_L(s^0)$.

Let $N \geqslant 3$, and be odd. The indices of the contours of S can be chosen such that $\operatorname{Int} \Gamma_1 \cap X \neq \varnothing$. Here X is the trajectory occurring among the conditions of the theorem. This time, $\partial_e = \Gamma_2 \circ \Gamma_4 \circ \cdots \circ \Gamma_{N-1}$ and $\partial_o = \Gamma_1 \circ \Gamma_3 \circ \cdots \circ \Gamma_N$. We have $k(\partial_e|\Omega_0) = 0$ and

$$0 \leqslant k(\partial_o|\Omega_0) = k(\Gamma_1 - \Gamma_N|\Omega_0) - k(\Gamma_1 \cap \Gamma_N|\Omega_0) < \vartheta$$

If γ' is a contour constructed in the same way as before, then $0 \leqslant k(\gamma'|\Omega_0) < \vartheta$ and $|\gamma'| \geqslant 4(Kc - 1) \geqslant L$. Therefore

$$k(\gamma'|\Omega_0)/|\gamma'| < \vartheta/4(Kc - 1) \leqslant c \tag{4.4}$$

which contradicts $0 \in \mathscr{P}_{L,c}(s^0)$. The second inequality of (4.4) comes from (4.1) and the fact that $L \geqslant 4$.

What remains is $N = 2$. If $0 \in \operatorname{Int}(\Gamma_1 \circ \Gamma_2)$, this does not differ from the case of even N. If $0 \notin \operatorname{Int}(\Gamma_1 \circ \Gamma_2)$ and Γ_1 is a contour with length less than ϑ, one can modify Γ_1 and get a Γ_1' such that $0 \in \operatorname{Int}(\Gamma_1' \circ \Gamma_2)$, $|\Gamma_1'| < \vartheta$, and $\operatorname{Int} \Gamma_1' \subset \operatorname{Int} \Gamma_1$. From the latter property it follows that $\rho(\Gamma_1', 0) \geqslant Kc/2$; if we put $\partial_e = \Gamma_2$ and $\partial_\sigma = \Gamma_1'$, the proof is the same as for odd N.

2. Now we turn to the proof of the second assertion of the theorem, i.e.,

$$\liminf_{M \to \infty} [|\mathscr{P}_4(s^0) \cap T_M|/|T_M|] > 0$$

One can easily show that this is true if

$$\liminf_{M \to \infty} [|\mathscr{P}_4(s^0) \cap \delta T_M|/|\delta T_M|] > 0 \tag{4.5}$$

where

$$\delta T_M = \{x \in \mathbb{Z}^2: \quad d(x, T_M) = 1\} = \{x \in \mathbb{Z}^2: \quad \|x\| = M + 1\}$$

The proof of (4.5) follows the same lines as that of the first part. Suppose (4.5) is not true; then there is a series of integers $0 < K_1 < K_2 < \cdots$ such that

$$\lim_{n \to \infty} [|\mathscr{P}_4(s^0) \cap \delta T_{K_n}|/|\delta T_{K_n}|] = 0$$

Let $\epsilon < c(c - 1/K)/4$, where K is given by (4.1). There is an $n_0 = n_0(\epsilon)$ such that $|\mathscr{P}_4(s^0) \cap \delta T_{K_n}| < 4\epsilon K_n$ if $n \geqslant n_0$. We fix an $n \geqslant n_0$ so large that $K_n \geqslant K$. Let $D_1, D_2, \ldots, D_m$ be the points of $\mathscr{P}_4(s^0) \cap \delta T_{K_n}$. The $\partial(D_i)$ are contours with positive energy, $\partial(D_i) \cap \partial(D_j) = \varnothing$, and

$$\sum_{i=1}^{m} |\partial(D_i)| = 4m < 16\epsilon K_n$$

For any $x \in \delta T_{K_n}$ we choose a $\Gamma(x)$; let this be a zero-energy contour around x if $L(x) > 4$ [see Eq. (2.6)] and be $\partial(D_i)$ if $x = D_i$. Starting with this set of contours, we repeat the procedure described above in part 1. When producing a minimal ring around the origin, some of the $\partial(D_i)$ contours may be ruled out; if that happens with all of them, the proof is reduced to that of the first assertion. If the minimal ring contains a $\partial(D_i)$, it will not be affected by the transformation [1(ii)]. Let $\{\Gamma_1,...,\Gamma_N\}$ be the finally obtained minimal ring and $k(\Gamma_1|\Omega_0) > 0$. This time,

$$\partial_e \equiv \mathop{\circ}_{\substack{i\,\text{even}\\ k(\Gamma_i|\Omega_0)=0}} \Gamma_i, \qquad \partial_o \equiv \Big(\mathop{\circ}_{\substack{i\,\text{odd}\\ k(\Gamma_i|\Omega_0)=0}} \Gamma_i \Big) \circ \Big(\mathop{\circ}_{j:k(\Gamma_j|\Omega_0)>0} \Gamma_j \Big)$$

One obtains that $k(\partial_e|\Omega_0) = 0$ and $0 \leqslant k(\partial_o|\Omega_0) \leqslant 16\epsilon K_n$. If

$$\gamma' \subset \bigcup_{x\in\delta T_{K_n}} \Gamma(x)$$

is a contour around the origin, constructed from ∂_e and ∂_o in the same way as in part 1(v), then $|\gamma'| \geqslant 4(K_n c - 1) \geqslant L$ and $0 \leqslant k(\gamma'|\Omega_0) \leqslant 16\epsilon K_n$. We therefore have

$$\frac{k(\gamma'|\Omega_0)}{|\gamma'|} \leqslant \frac{4\epsilon K_n}{K_n c - 1} < c$$

which contradicts $0 \in P_{L,c}(s^0)$.

5. DISCUSSION

The search for models of spin-glasses has given rise to a large body of work on the thermodynamic properties of systems with a frustration potential. We do not wish to review this field, nor to discuss the properties of spin-glasses, but only mention that many works conclude with some negative statement concerning the existence of a suitable frustration model in two (and probably in three) dimensions. The two-dimensional "odd model" proposed by Villain[6] is a model without a phase transition; André *et al.*,[7] in a study with periodic Ising frustration potentials in $d = 2$, noticed that their system, though undergoing a phase transition, always contained infinite connected sets of ferro- or antiferromagnetically ordered spins in the ground states. In "true" models of spin-glasses the interactions J_{xy} are considered to be random variables. A series of Monte Carlo studies with such potentials[8,9] also suggested the absence of spin-glass behavior.

The motivation for the present work was to throw some light on the background of this failure. Though the comparison of the present study with the above work is not immediate, we think that some of the features of our results are rather suggestive. If spin-glasses have to be described by phase transition models, the dominance of free blocks, i.e., zero-energy contours, can make them different from magnetically ordered materials. According to our second theorem, however, zero-energy contours do not play a dominant role in two-dimensional phase transition models.

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